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Realisations of the Representations of Para-Fermi Algebras—Part IV: Fock Spaces of Para-Bose and Para-Fermi Operators

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Abstract

The representations of the para-Fermi algebra in the Fock spaces of para-Bose and para-Fermi operators are constructed. The unitary equivalence of different representations is proved. The Bardeen–Cooper–Schrieffer pair creation and annihilation operators and the four fermion interaction appear as particular realisations of the para-Fermi algebra. The para-Fermi algebra representations in quantum mechanics are discussed.

1. *Introduction*

In Parts I, II and III of this series (Kademova, 1970; Kademova & Kálnay, 1970; Kademova & Kraev, 1970) making use of the isomorphic mappings of the para-Fermi algebra into the bilinear polynomials of the parafield operators, the problem of realising the representations of this algebra in the Fock space of Bose or Fermi operators has been studied.

In Section 2 of the present article we show that all the representations of the para-Fermi algebra with two generators can be realised in the single particle subspaces $H_{q\epsilon}^{p+1}$ of the Fock space of $p+1$ ($p=1,2,\ldots$) para-Bose or para-Fermi operators of arbitrary order of parastatistics $q=1,2,\ldots$. The representations realised in these spaces are unitarily equivalent for fixed p and arbitrary q and ϵ. The parastatistics of the induced transformations depends only on the number $p+1$ of the operators and does not depend on their parastatistics and their type. We show also that the representations induced in $H_{q\epsilon}^{p+1}$ are unitarily equivalent to the representations induced in the space H_p (p-particle subspace of the Fock space of two Bose operators). The representations induced in the one-particle subspace of the Fock space of infinitely many parafield operators form a Bose algebra.

In Section 3 the results of Greenberg and Messiah connected with the commutation relations between different fields are reviewed. Using them we show in Section 4 that the four fermion interaction Hamiltonian appears as a particular case of the high-order realisations of the para-Fermi algebra. The Bardeen–Cooper–Schrieffer pair creation and annihilation operators in the theory of superconductivity are para-Fermi algebra generators as well.

Finally, in Section 5 we discuss the meaning of the para-Fermi algebra representations in quantum mechanical terms.

2. *Realisations of the Representations of the para-Fermi Algebra in the Fock Spaces of para-Bose and para-Fermi Operators*

We start with the Green isomorphic mappings of an arbitrary para-Fermi algebra with two generators of order of parastatistics p into the bilinear polynomials of $p+1$ para-Fermi or para-Bose operators of parastatistics q, which were defined in Parts I and III of this series of articles.

$$i^p_{q\epsilon}:\quad \begin{aligned} F^p &\to \mathscr{F}^p_{q\epsilon} = \tfrac{1}{2}(F^p)_{i\,i+1}\,[\overset{+}{a}_i, a_{i+1}]_\epsilon \\ \overset{+}{F^p} &\to \overset{+}{\mathscr{F}}{}^p_{q\epsilon} = (\mathscr{F}^p_{q\epsilon})^+ \end{aligned} \tag{2.1}$$

where $\overset{+}{a}_i$, a_j, i, $j = 1,\ldots,p+1$, stand for para-Bose operators of parastatistics q for positive ϵ and for para-Fermi ones for negative ϵ.

We shall find all the representations of the para-Fermi algebra in the space $H^{p+1}_{q\epsilon}$, $p = 1,2,\ldots$, of $p+1$ para-Bose or para-Fermi operators of order of parastatistics q.

For this purpose we embed, using the Green Ansatz (Green, 1953)

$$\begin{aligned} a_i &= \sum_{\alpha=1}^{q} a_i^\alpha \\ \overset{+}{a}_i &= \sum_{\alpha=1}^{q} \overset{+}{a}{}_i^\alpha \end{aligned} \tag{2.2}$$

the algebra of the parafield operators $\overset{+}{a}_i$, a_j, i, $j = 1,\ldots,p+1$, into the algebra $\mathscr{U}(p+1,q,\epsilon)$ (Kademova & Palev, 1970) of the quasifield operators $\overset{+}{a}{}_i^\alpha$, a_j^β, $i, j = 1,\ldots,p+1$, α, $\beta = 1,\ldots,q$, for which the relations

$$\begin{aligned} &[a_i^\alpha, \overset{+}{a}{}_j^\alpha]_{-\epsilon} = \delta_{ij}, \qquad [a_i^\alpha, a_j^\alpha]_{-\epsilon} = [\overset{+}{a}{}_i^\alpha, \overset{+}{a}{}_j^\alpha]_{-\epsilon} = 0 \\ &[a_i^\alpha, \overset{+}{a}{}_j^\beta]_\epsilon = [a_i^\alpha, a_j^\beta]_\epsilon = [\overset{+}{a}{}_i^\alpha, \overset{+}{a}{}_j^\beta]_\epsilon = 0 \qquad (\alpha \neq \beta) \end{aligned} \tag{2.3}$$

hold.

Let us now consider the transformations induced by the operators $\mathscr{F}^p_{q\epsilon}$, $\overset{+}{\mathscr{F}}{}^p_{q\epsilon}$ in the space $H^{p+1}_{q\epsilon}$ spanned on the vectors $\overset{+}{a}_k|0\rangle$, $k = 1,\ldots,p+1$ (the operators $\overset{+}{a}_k$ are para-Bose or para-Fermi ones of order of parastatistics q).

$$\mathscr{F}^p_{q\epsilon}\overset{+}{a}_k|0\rangle = \tfrac{1}{2}(F^p)_{i\,i+1}\,[\overset{+}{a}_i, a_{i+1}]_\epsilon\,\overset{+}{a}_k|0\rangle$$

Using the Green Ansatz (2.2) this can be put into the form

$$\begin{aligned}\mathscr{F}^{p}_{q\epsilon}\overset{+}{a}_k|0\rangle &= \tfrac{1}{2}(F^p)_{i\,i+1}\sum_{\alpha,\beta,\gamma=1}^{q}[\overset{+}{a}_i{}^{\alpha}, a^{\gamma}_{i+1}]_\epsilon\, \overset{+}{a}_k{}^{\beta}|0\rangle\\ &= \tfrac{1}{2}(F^p)_{i\,i+1}\sum_{\alpha,\beta=1}^{q}[\overset{+}{a}_i{}^{\alpha}, a^{\alpha}_{i+1}]_\epsilon\, \overset{+}{a}_k{}^{\beta}|0\rangle\end{aligned}$$

Here the relations (2.3) were used. After adding and subtracting some terms we obtain

$$\begin{aligned}\mathscr{F}^{p}_{q\epsilon}\overset{+}{a}_k|0\rangle = \tfrac{1}{2}(F^p)_{i\,i+1}\Bigg\{&\sum_{\alpha=1}^{q}\overset{+}{a}_i{}^{\alpha}[a^{\alpha}_{i+1},\overset{+}{a}_k{}^{\alpha}]_{-\epsilon} + \epsilon\sum_{\alpha=1}^{q}\overset{+}{a}_i{}^{\alpha}\overset{+}{a}_k{}^{\alpha}a^{\alpha}_{i+1} +\\ &+\sum_{\substack{1\\ \alpha\neq\beta}}^{q}\overset{+}{a}_i{}^{\alpha}[a^{\alpha}_{i+1},\overset{+}{a}_k{}^{\beta}]_{\epsilon} - \epsilon\sum_{\substack{1\\ \alpha\neq\beta}}^{q}\overset{+}{a}_i{}^{\alpha}\overset{+}{a}_k{}^{\beta}a^{\alpha}_{i+1} +\\ &+\epsilon\sum_{\alpha,\beta=1}^{q}[a^{\alpha}_{i+1},\overset{+}{a}_i{}^{\alpha}]_{-\epsilon}\overset{+}{a}_k{}^{\beta} + \epsilon^2\sum_{\alpha=1}^{q}\overset{+}{a}_i{}^{\alpha}[a^{\alpha}_{i+1},\overset{+}{a}_k{}^{\alpha}]_{-\epsilon} +\\ &+\epsilon^3\sum_{\alpha=1}^{q}\overset{+}{a}_i{}^{\alpha}\overset{+}{a}_k{}^{\alpha}a^{\alpha}_{i+1} + \epsilon^2\sum_{\substack{1\\ \alpha\neq\beta}}^{q}\overset{+}{a}_i{}^{\alpha}[a^{\alpha}_{i+1},\overset{+}{a}_k{}^{\beta}]_{\epsilon} -\\ &-\epsilon^3\sum_{\substack{1\\ \alpha\neq\beta}}^{q}\overset{+}{a}_i{}^{\alpha}\overset{+}{a}_k{}^{\beta}a^{\alpha}_{i+1}\Bigg\}|0\rangle\end{aligned}$$

Now using the relations (2.3) and the fact that the operators $a_i{}^{\alpha}$, $i = 1,\ldots,p+1$, $\alpha = 1,\ldots,q$, acting on the vacuum $|0\rangle$ give zero we obtain that only the first and the sixth terms give non-zero contribution. Since $\epsilon^2 = 1$ we finally get

$$\mathscr{F}^{p}_{q\epsilon}\overset{+}{a}_k|0\rangle = (F^p)_{k-1\,k}\overset{+}{a}_{k-1}|0\rangle \tag{2.4a}$$

In a similar way one gets

$$\overset{+}{\mathscr{F}}{}^{p}_{q\epsilon}\overset{+}{a}_k|0\rangle = (\overset{+}{F}{}^p)_{k+1\,k}\overset{+}{a}_{k+1}|0\rangle \tag{2.4b}$$

So the space $H^{p+1}_{q\epsilon}$ is invariant under the transformations (2.4a) and (2.4b). Moreover, these transformations form a para-Fermi algebra of parastatistics p. The vector $\overset{+}{a}_1|0\rangle$ plays a role of a vacuum for this algebra and

$$\mathscr{F}^{p}_{q\epsilon}\overset{+}{\mathscr{F}}{}^{p}_{q\epsilon}\overset{+}{a}_1|0\rangle = p\overset{+}{a}_1|0\rangle$$

Thus for arbitrary fixed q and ϵ ($q = 1,2,\ldots,\epsilon = \pm$) the vectors $\overset{+}{a}_i|0\rangle$, $i = 1,\ldots,p+1$, span a representation of the para-Fermi algebra of parastatistics p.

A question arises whether the representations for two different pairs q, ϵ and q', ϵ' are essentially different.

Let us consider the spaces $H_{q\epsilon}^{p+1}$ and $H_{q'\epsilon'}^{p+1}$ spanned on the vectors $\overset{+}{a}_i(q,\epsilon)|0\rangle$ and $\overset{+}{a}_i(q',\epsilon')|0\rangle$, $i=1,\ldots,p+1$, respectively (we use the indices q, ϵ and q', ϵ' in order to avoid confusion). And let us define an operator

$$\mathscr{U}=\sum_{i=1}^{p+1}\overset{+}{a}_i(q',\epsilon')\,a_i(q,\epsilon).\frac{1}{\sqrt{(qq')}} \tag{2.5}$$

This operator maps the space $H_{q\epsilon}^{p+1}$ onto the space $H_{q'\epsilon'}^{p+1}$. More precisely, it transforms the vector $\overset{+}{a}_i(q,\epsilon)|0\rangle\in H_{q\epsilon}^{p+1}$ into the vector $\overset{+}{a}_i(q',\epsilon')\in H_{q'\epsilon'}^{p+1}$, $i=1,\ldots,p+1$. This can be proved in a way similar to that used for obtaining the formula (2.4a).

The inverse transformation mapping $H_{q'\epsilon'}^{p+1}$ onto $H_{q\epsilon}^{p+1}$ is

$$\mathscr{U}^{-1}=\sum_{i=1}^{p+1}\overset{+}{a}_i(q,\epsilon)\,a_i(q',\epsilon').\frac{1}{\sqrt{(qq')}} \tag{2.6}$$

So the transformation $\mathscr{U}$ is a unitary one.

Thus, the fact that the representations of the para-Fermi algebra realised in the spaces $H_{q\epsilon}^{p+1}$ and $H_{q'\epsilon'}^{p+1}$ are unitarily equivalent is proved, which is in agreement with the theorem of Greenberg and Messiah (Greenberg & Messiah, 1965b).

In Section 1 we proved that all the representations of the para-Fermi algebra with two generators can be realised in the Fock space of two Bose operators. In the p Bose particle subspace $H_p\subset\mathscr{H}_2{}^1$ an irreducible representation of the para-Fermi algebra of order of parastatistics p is realised.

We shall show that there exists a unitary transformation which maps the subspace H_p onto the space $H_{q\epsilon}^{p+1}$ (q, ϵ-arbitrary), i.e. that the induced transformations in these spaces form unitarily equivalent irreducible representations of the para-Fermi algebra of parastatistics p.

Let us remember that the space H_p is spanned on the vectors

$$\frac{(\overset{+}{b}_1)^\alpha(\overset{+}{b}_2)^\beta}{\sqrt{(\alpha!\,\beta!)}}|0\rangle$$

where $\alpha+\beta=p$.

Then

$$\mathscr{U}_1=\sum_{i=1}^{p+1}\frac{(\overset{+}{b}_1)^{p-i+1}(\overset{+}{b}_2)^{i-1}}{\sqrt{[(p-i+1)!\,(i-1)!]}}a_i.\frac{1}{\sqrt{q}} \tag{2.7}$$

is a transformation mapping the space $H_{q\epsilon}^{p+1}$ (here we do not attach the indices q, ϵ to a_i, since no confusion can arise) onto the space H_p. More precisely the vector $\overset{+}{a}_i|0\rangle\in H_{q\epsilon}^{p+1}$ is mapped into

$$\frac{(\overset{+}{b}_1)^{p-i+1}(\overset{+}{b}_2)^{i-1}}{\sqrt{[(p-i+1)!\,(i-1)!]}}|0\rangle\in H_p$$

The inverse transformation is

$$\mathscr{U}_1^{-1} = \sum_{i=1}^{p+1} \overset{+}{a}_i \frac{(b_2)^{i-1}(b_1)^{p-i+1}}{\sqrt{[(i-1)!(p-i+1)!]}} \cdot \frac{1}{\sqrt{q}} \tag{2.8}$$

Thus the unitary equivalence of the representations of the para-Fermi algebra in the spaces defined above is shown.

With respect to the transformations induced by the para-Fermi algebra the spaces $H_{q\epsilon}^{p+1}$ (for p fixed and q, ϵ arbitrary) are indistinguishable between themselves, and each of them is indistinguishable from the space H_p. In these spaces unitarily equivalent transformations of the para-Fermi algebra are realised.

Consider now the high order limit $p \to \infty$ of the mappings $i_{q\epsilon}^p$. It has been proved by Greenberg and Messiah (Greenberg & Messiah, 1965a) that the high order limit of the para-Fermi algebra is a Bose algebra, i.e. the para-Fermi operators

$$B = \lim_{p\to\infty} \frac{F^p}{\sqrt{p}}, \qquad \overset{+}{B} = \lim_{p\to\infty} \frac{\overset{+}{F^p}}{\sqrt{p}}$$

satisfy the Bose commutation relations. Therefore, the operators defined through the mappings

$$\lim_{p\to\infty} i_{q\epsilon}^p : \begin{array}{l} B \to \mathscr{B}_{q\epsilon} = \lim\limits_{p\to\infty} \dfrac{\mathscr{F}_{q\epsilon}^p}{\sqrt{p}} = (B)_{ij}\,[\overset{+}{a}_i, a_j]_\epsilon \\ \overset{+}{B} \to \overset{+}{\mathscr{B}}_{q\epsilon} = (\mathscr{B}_{q\epsilon})^+ \end{array} \tag{2.9}$$

form obviously a Bose algebra in the space

$$\lim_{p\to\infty} H_{q\epsilon}^{p+1}$$

It is clear that the representations of the Bose algebra realised in the spaces

$$\lim_{p\to\infty} H_p \subset \mathscr{H}_2{}^1 \qquad \text{and} \qquad \lim_{p\to\infty} H_{q\epsilon}^{p+1}$$

are unitarily equivalent.

3. *Commutation Relations Between Different Parafields*

Here we briefly review the results of Greenberg & Messiah (1965b) concerning the commutation relations between different parafields. It has been shown by them that the most general commutation relations between different parafields should be trilinear. In order to limit the choice of the trilinear commutation relations the authors adopt requirements such that the commutation relations for one parafield be a special case of them. They demand: (i) the left-hand side of the commutation relations must have the form $[[a,b]_\epsilon,c]_\eta$ with ϵ, $\eta = \pm$, and the right hand side must be linear;

(ii) when the pair $[a,b]_\epsilon$ refers to one and the same field ($\epsilon = \pm$ for para-Bose and para-Fermi cases respectively) it must commute with $c(\eta = -)$ if c refers to another field; (iii) these relations must be satisfied by the ordinary Bose and Fermi fields.

In the case of two parafields a_i and b_j, $i, j = 1,\dots,n$, using the conditions (i)–(iii) one gets the following commutation relations. For one of them a_i they are of the known type

$$\begin{aligned} [\tfrac{1}{2}[\overset{+}{a}_i, a_j]_{\epsilon^a}, \overset{+}{a}_k]_- &= \delta_{jk}\overset{+}{a}_i \\ [[a_i, a_j]_{\epsilon^a}, a_k]_- &= 0 \end{aligned} \tag{3.1}$$

and the same for b_i. By ϵ^a and ϵ^b the type of the fields a_i and b_j, respectively, is denoted.

The commutation relations involving both fields are

$$[[\overset{+}{a}_i, a_j]_{\epsilon^a}, b_k]_- = 0 \tag{3.2}$$

$$[[a_i, a_j]_{\epsilon^a}, b_k]_- = 0 \tag{3.3}$$

$$[[\overset{+}{a}_i, \overset{+}{a}_j]_{\epsilon^a}, b_k]_- = 0 \tag{3.4}$$

$$[\tfrac{1}{2}[b_i, \overset{+}{a}_j]_\eta, a_k]_{-\eta\epsilon^a} = -\epsilon^a\delta_{jk} b_i \tag{3.5}$$

$$[\tfrac{1}{2}[a_i, b_j]_\eta, \overset{+}{a}_k]_{-\eta\epsilon^a} = \eta\delta_{ik} b_j \tag{3.6}$$

$$[[a_i, \overset{+}{a}_j]_{\epsilon^a}, \overset{+}{b}_k]_- = 0 \tag{3.7}$$

$$[\tfrac{1}{2}[\overset{+}{b}_i, a_j]_\eta, \overset{+}{a}_k]_{-\eta\epsilon^a} = \delta_{jk}\overset{+}{b}_i \tag{3.8}$$

$$[\tfrac{1}{2}[\overset{+}{a}_i, \overset{+}{b}_j]_\eta, a_k]_{-\eta\epsilon^a} = -\eta\epsilon^a\delta_{ik}\overset{+}{b}_j \tag{3.9}$$

$$[[b_i, a_j]_\eta, a_k]_{-\eta\epsilon^a} = 0 \tag{3.10}$$

$$[[b_i, \overset{+}{a}_j]_\eta, \overset{+}{a}_k]_{-\eta\epsilon^a} = 0 \tag{3.11}$$

The index $\eta = \pm$ takes on one and the same value for all the commutation relations (for $\eta = +$ the two fields are relative para-Bose and for $\eta = -$ they are relative para-Fermi).

The commutation relations (3.2)–(3.11) are obtained by using the conditions (i)–(iii), the hermitian conjugation and the generalised Jacobi identity

$$[[a,b]_\epsilon, c]_- + [[c,a]_\eta, b]_{-\eta\epsilon} + \eta\epsilon[[b,c]_\eta, a]_{-\eta\epsilon} = 0 \tag{3.12}$$

All the other commutation relations are obtained by interchanging a_i and b_i.

It can be shown as in the case of a single parafield that using the Green Ansatz the algebra of the parafield operators can be isomorphically embedded into the algebra of the quasifield operators a_i^{α}, b_j^{β}.

The operators a_i^{α}, b_j^{β} satisfy the commutation relations (2.3) of the quasifield operators and the following relative commutation relations

$$\begin{aligned} &[a_i^{\alpha}, b_j^{\alpha}]_{-\eta} = 0 \quad , \quad [a_i^{\alpha}, \overset{+}{b}_j{}^{\alpha}]_{-\eta} = 0 \\ &[a_i^{\alpha}, b_j^{\beta}]_{\eta} = 0, \qquad [a_i^{\alpha}, \overset{+}{b}_j{}^{\beta}]_{\eta} = 0 \qquad (\alpha \neq \beta) \end{aligned} \tag{3.13}$$

The index η takes on the same values as in (3.2)–(3.11) and determines the relative type of the quasifield operators a_i^{α} and b_j^{β}.

It has been proved that all Fock representations of the commutation relations (3.1)–(3.11) are given by the Green Ansatz and are determined by the existence of a no-particle vector $|0\rangle$ and the conditions

$$\begin{aligned} a_i \overset{+}{a}_j |0\rangle &= p\delta_{ij} |0\rangle \\ b_i \overset{+}{b}_j |0\rangle &= p\delta_{ij} |0\rangle \\ a_i \overset{+}{b}_j |0\rangle &= 0 \\ b_i \overset{+}{a}_j |0\rangle &= 0 \end{aligned} \tag{3.14}$$

The commutation relations for more than two fields follow from the results of Greenberg and Messiah. Let a_i, b_j, c_k, be different parafields and η^{ab}, η^{ac}, η^{bc} their relative types. Then

$$[[a_i, b_j]_{\eta^{ab}}, c_k]_- = 0 \tag{3.15}$$

and all the other commutation relations involving three of the fields are also zeros. This can be proved by the use of the Green Ansatz and the commutation relations (3.13).

4. *Some Quantum Theory Operators as Generators of para-Fermi Algebras*

Here we show that as a consequence of the para-Fermi algebra realisations the Bardeen–Cooper–Schrieffer (BCS) pair creation and annihilation operators in the theory of the superconductivity and the four fermion interaction appear as elements of the para-Fermi algebra.

4.1. *BCS Operators as para-Fermi Algebra Generators*

A general method for realising para-Fermi algebra generators as high-order polynomials of parafield operators of one and the same field has been given in (Kademova, 1970b). From these realisations it follows straightforwardly that the entities

$$\begin{aligned}\mathscr{F} &= f_1^{1/2} f_2^{-1/2}\\ \overset{+}{\mathscr{F}} &= \overset{+}{f}{}_2^{-1/2} f_1^{1/2}\end{aligned} \tag{4.1.1}$$

generate a para-Fermi algebra with $f_1^{1/2}, f_2^{-1/2}$ being Fermi operators. The operator $\overset{+}{f}{}_1^{1/2}$ creates a Fermi particle with spin $\frac{1}{2}$ in the level 1 and the operator $\overset{+}{f}{}_2^{-1/2}$ creates a Fermi particle with spin $-\frac{1}{2}$ in the level 2. (In Kademova (1970b) it is to be understood that the indices $\frac{1}{2}$ and $-\frac{1}{2}$ are included in the subscripts.)

In the space X spanned on the vectors $|0\rangle$ and $\overset{+}{f}{}_2^{-1/2}\overset{+}{f}{}_1^{1/2}|0\rangle$ the operators (4.1.1) induce transformations which form a Fermi algebra.

If in (4.1.1) we replace the indices 1 and 2 by k and $-k$ we get the pair creation and annihilation operators in the BCS theory of superconductivity. Consider the BCS pair creation and annihilation operators $\overset{+}{\mathscr{F}}_k$ and $\mathscr{F}_k$ defined through $\overset{+}{f}{}_k^{1/2}, f_k^{1/2}, \overset{+}{f}{}_{-k}^{-1/2}, f_{-k}^{-1/2}$ for all possible values of $k = 1,\ldots,n$. Each pair form a para-Fermi algebra with two generators and all the pairs define a direct sum of para-Fermi algebras with two generators each. The transformations induced in the space spanned on the vectors

$$|0\rangle, \prod_{k=1}^{i} \overset{+}{f}{}_{-k}^{-1/2}\overset{+}{f}{}_k^{1/2}\,|0\rangle$$

$i = 1,\ldots,n$, contain a para-Fermi algebra with two generators of parastatistics n. In the case of $n \to \infty$ they contain a Bose algebra with two generators.

It is interesting to remark that the operators

$$\mathscr{F} = \overset{+}{f}{}_1^{1/2} f_2^{-1/2}, \qquad \overset{+}{\mathscr{F}} = \overset{+}{f}{}_2^{-1/2} f_1^{1/2} \tag{4.1.2}$$

which are a particular case of the realisations (2.1) for $p = 1$, $q = 1$ and $\epsilon = -$, induce transformations in the space X_1: $\overset{+}{f}{}_1^{1/2}|0\rangle, \overset{+}{f}{}_2^{-1/2}|0\rangle$, which form a representation unitarily equivalent to the representation of the operators (4.1.1) in the space X. The unitary transformation is

$$\mathscr{U}_2 = f_1^{1/2} - \overset{+}{f}{}_1^{1/2}$$

4.2. *Para-Fermi Algebra Realisations by means of Different Parafields*

Let us now construct some particular realisations of the para-Fermi algebra by means of parafield operators corresponding to different parafields.

For this purpose we start with the entities

$$\begin{aligned}\overset{+}{\mathscr{F}} &= \tfrac{1}{2}[\overset{+}{a}, b]_\eta \\ \mathscr{F} &= \tfrac{1}{2}[\overset{+}{b}, a]_\eta\end{aligned} \tag{4.2.1}$$

where a and b both are para-Fermi or para-Bose operators of different parafields which are relative para-Fermi or para-Bose, i.e. $\epsilon^a = \epsilon^b = \epsilon = \pm$, $\eta = \pm$.

Let us show that the entities (4.2.1) generate a para-Fermi algebra. Using (3.5) and (3.8) we obtain

$$\begin{aligned}[\overset{+}{\mathscr{F}},\mathscr{F}]_- &= \tfrac{1}{4}[[\overset{+}{a},b]_\eta, \quad [\overset{+}{b},a]_\eta]_- = \tfrac{1}{2}\{[\tfrac{1}{2}[\overset{+}{a},b]_\eta,\overset{+}{b}]_{-\epsilon\eta}\, a + \\ &\quad + \epsilon\eta\overset{+}{b}[\tfrac{1}{2}[\overset{+}{a},b]_\eta,a]_{-\epsilon\eta} + \eta[\tfrac{1}{2}[\overset{+}{a},b]_\eta,a]_{-\epsilon\eta}\,\overset{+}{b} + \epsilon a[\tfrac{1}{2}[\overset{+}{a},b]_\eta,\overset{+}{b}]_{-\epsilon\eta}\} \\ &= \tfrac{1}{2}\{\overset{+}{a}a - \overset{+}{b}b - \epsilon b\overset{+}{b} + \epsilon a\overset{+}{a}\} \\ &= \tfrac{1}{2}\{[\overset{+}{a},a]_\epsilon - [\overset{+}{b},b]_\epsilon\}\end{aligned}$$

Further, using (3.1), (3.2) and (3.7) we get

$$\begin{aligned}[\tfrac{1}{2}[\overset{+}{\mathscr{F}},\mathscr{F}]_-,\overset{+}{\mathscr{F}}]_- &= \tfrac{1}{4}\{[\tfrac{1}{2}[\overset{+}{a},a]_\epsilon,[\overset{+}{a},b]_\eta]_- - [\tfrac{1}{2}[\overset{+}{b},b]_\epsilon,[\overset{+}{a},b]_\eta]_-\} \\ &= \tfrac{1}{4}\{[\tfrac{1}{2}[\overset{+}{a},a]_\epsilon,\overset{+}{a}]_-\, b + \eta b[\tfrac{1}{2}[\overset{+}{a},a]_\epsilon\, \overset{+}{a}]_- - \\ &\quad - \overset{+}{a}[\tfrac{1}{2}[\overset{+}{b},b]_\epsilon,b]_- - \eta[\tfrac{1}{2}[\overset{+}{b},b]_\epsilon,b]_-\,\overset{+}{a}\} \\ &= \tfrac{1}{4}\{\overset{+}{a}b + \eta b\overset{+}{a} + \overset{+}{a}b + \eta b\overset{+}{a}\} = \tfrac{1}{2}[\overset{+}{a},b]_\eta = \overset{+}{\mathscr{F}}\end{aligned}$$

So we have

$$[\tfrac{1}{2}[\overset{+}{\mathscr{F}},\mathscr{F}]_-,\overset{+}{\mathscr{F}}]_- = \overset{+}{\mathscr{F}}$$

Obviously the relation

$$[[\mathscr{F},\mathscr{F}]_-,\mathscr{F}]_- = 0$$

holds.

Thus we have shown that $\mathscr{F}$ and $\overset{+}{\mathscr{F}}$ generate a para-Fermi algebra with two generators.

Let us now define by analogy with (4.2.1) the operators $\overset{+}{G}$ and G

$$\overset{+}{G} = \tfrac{1}{2}[\overset{+}{c},d]_{\eta'}, \qquad G = \tfrac{1}{2}[\overset{+}{d},c]_{\eta'} \tag{4.2.2}$$

As in (4.2.1) both operators c and d are para-Fermi or para-Bose, $\epsilon^c = \epsilon^d = \epsilon' = \pm$, and are relative para-Fermi or para-Bose, $\eta' = \pm$. The operators $\overset{+}{G}$ and G generate a para-Fermi algebra with two generators, too.

Consider now the commutators of the para-Fermi operators (4.2.1) and (4.2.2). Using the relations (3.15) we obtain

$$[\overset{+}{\mathscr{F}}, \overset{+}{G}]_- = \tfrac{1}{4}[[\overset{+}{a}, b]_\eta, [\overset{+}{c}, d]_{\eta'}]_- = \tfrac{1}{4}\{[[\overset{+}{a}, b]_\eta, \overset{+}{c}]_- d + \overset{+}{c}[[\overset{+}{a}, b]_\eta, d]_- + \\ + \eta'[[\overset{+}{a}, b]_\eta, d]_- \overset{+}{c} + \eta' d[[\overset{+}{a}, b]_\eta, \overset{+}{c}]_-\} = 0$$

Similarly, all the other commutators are zeros.

Using (4.2.1) and (4.2.2) we finally construct the entities

$$\begin{aligned} \mathscr{L} &= \tfrac{1}{2}[\overset{+}{\mathscr{F}}, G]_+ = \tfrac{1}{8}[[\overset{+}{a}, b]_\eta, [\overset{+}{d}, c]_{\eta'}]_+ \\ \overset{+}{\mathscr{L}} &= \tfrac{1}{2}[\overset{+}{G}, \mathscr{F}]_+ = \tfrac{1}{8}[[\overset{+}{c}, d]_{\eta'}, [\overset{+}{b}, a]_\eta]_+ \end{aligned} \tag{4.2.3}$$

4.3. *Four Fermion Interaction as an Element of the para-Fermi Algebra*

The entities $\mathscr{L}$ and $\overset{+}{\mathscr{L}}$ have been constructed by means of arbitrary different parafields. Let us consider the particular case when a, b, c, d are Fermi operators. Then the operators

$$\begin{aligned} \mathscr{L} &= \overset{+}{\mathscr{F}} G = \overset{+}{a} b \overset{+}{d} c \\ \overset{+}{\mathscr{L}} &= \overset{+}{G} \mathscr{F} = \overset{+}{c} d \overset{+}{b} a \end{aligned} \tag{4.3.1}$$

are generators of a para-Fermi algebra. This can be readily checked. Making use of the fact that the operators (4.2.1) and (4.2.2) commute we get

$$\begin{aligned} [[\overset{+}{\mathscr{L}}, \mathscr{L}]_-, \overset{+}{\mathscr{L}}]_- &= [[\overset{+}{G}\mathscr{F}, \overset{+}{\mathscr{F}} G]_-, \overset{+}{G}\mathscr{F}]_- \\ &= 2\overset{+}{G}G\overset{+}{G}\mathscr{F}\overset{+}{\mathscr{F}}\mathscr{F} - \overset{+}{G}\overset{+}{G}G\mathscr{F}\mathscr{F}\overset{+}{\mathscr{F}} - G\overset{+}{G}\overset{+}{G}\overset{+}{\mathscr{F}}\mathscr{F}\mathscr{F} \end{aligned}$$

Now using the particular realisations of $\mathscr{F}$, $\overset{+}{\mathscr{F}}$, G, $\overset{+}{G}$ and $\mathscr{L}$, $\overset{+}{\mathscr{L}}$ by means of Fermi operators we obtain

$$[\tfrac{1}{2}[\overset{+}{\mathscr{L}}, \mathscr{L}]_-, \overset{+}{\mathscr{L}}]_- = \overset{+}{\mathscr{L}}$$

Thus it is shown that $\mathscr{L}$ and $\overset{+}{\mathscr{L}}$ generate a para-Fermi algebra.

Consider the transformations induced by the operators (4.3.1) in the space spanned by the vectors $\overset{++}{da}|0\rangle$ and $\overset{++}{cb}|0\rangle$

$$\begin{aligned}\overset{+}{\mathscr{L}}\overset{++}{da}|0\rangle &= \overset{++}{cb}|0\rangle\\ \overset{+}{\mathscr{L}}\overset{++}{cb}|0\rangle &= 0\\ \mathscr{L}\overset{++}{cb}|0\rangle &= \overset{++}{da}|0\rangle\\ \mathscr{L}\overset{++}{da}|0\rangle &= 0\end{aligned} \tag{4.3.2}$$

The operators $\mathscr{L}$ and $\overset{+}{\mathscr{L}}$ form a Fermi algebra in this space.

The four fermion interaction Hamiltonian can be considered as a linear combination of operators of the type (4.3.1). If we attach the indices i, j, k, l to the operators a, b, c, d, the four fermion interaction Hamiltonian can be put in the form

$$\mathscr{H} = \sum_{i,j,k,l} \mathscr{T}_{ijkl}\, \overset{+}{a}_i b_j \overset{+}{d}_k c_l + \text{h.c.} \tag{4.3.3}$$

where $\mathscr{T}_{ijkl}$ are coefficients. Therefore, $\mathscr{H}$ is a linear combination of the generators of para-Fermi algebras with two generators each

$$\mathscr{L}_{ijkl} = \overset{+}{a}_i b_j \overset{+}{d}_k c_l, \qquad \overset{+}{\mathscr{L}}_{ijkl} = \overset{+}{c}_l d_k \overset{+}{b}_j a_i$$

The set of indices i, j, k, l label the para-Fermi algebras.

5. *Para-Fermi Algebra Invariance*

In the previous sections we have constructed the representations of the para-Fermi algebra in terms of quantum field operators. Let us now discuss the induced transformations in the case of quantum mechanics.

Consider a system of identical non-interacting particles which can occupy a number of different states. As a consequence of the permutation invariance the Hartree wave functions for the system can be divided into equivalence classes. Two wave functions are said to be equivalent if they can be reached one from another by a permutation. In each class a representation of the permutation group is realised which is generally reducible. By a convenient choice of the basis one can reduce this representation space. In the subspaces of the symmetrical (antisymmetrical) functions, describing Bose (Fermi) particles, irreducible representations of the permutation group are realised. In the case of only two identical particles these two subspaces constitute all the space, while in the case of more particles there are other higher dimensional subspaces. Operators N_{ij} can be introduced (the indices i and j denote the possible states of the particles) which act transitively in the space of the equivalence classes. The operators N_{ij} applied on a function of a given class transform it into a function of such a class that the number of the particles in the state j is less by one and in the state i more by one.

In quantum mechanics only symmetrical or antisymmetrical wave functions are used in accordance with the symmetrisation postulate. In this case, because of the isomorphism between the quantum mechanical and the quantum field theory pictures, the operators N_{ij} can be put into the form $N_{ij} = \overset{+}{a}_i a_j$, where $\overset{+}{a}_i$ and a_j are creation and annihilation Bose or Fermi operators. It is seen from (2.1) that the para-Fermi algebra generators can be expressed as linear combinations of the operators $N_{ij} = \overset{+}{a}_i a_j$. In quantum mechanical terms the representations of the para-Fermi algebra are realised in the space of the equivalence classes of symmetrical or antisymmetrical functions. In the case of n Bose particles and two states the space is transformed according to an irreducible representation of the para-Fermi algebra with two generators of parastatistics n. In the case of only one Bose or Fermi particle and $n + 1$ states each class contains only one function and the whole space is transformed under an irreducible representation of the para-Fermi algebra with two generators of parastatistics n. For m Bose or Fermi particles and n states such a simple structure of the space of the equivalence classes does not exist. The representations of the para-Fermi algebra are generally reducible.

The consequences of the para-Fermi algebra invariance will be studied further.

References

Green, H. S. (1953). *Physical Review*, **90**, 270.

Greenberg, O. W. and Messiah, A. M. L. (1965a). *Journal of Mathematics and Physics*, **6**, 500.

Greenberg, O. W. and Messiah, A. M. L. (1965b). *Physical Review*, **138**, 1155.

Kademova, K. V. (1970a). *International Journal of Theoretical Physics*, Vol. 3, No. 2, p. 109.

Kademova, K. V. (1970b). High-Order Realisations of the para-Fermi Algebra with Parafield Operators. *International Journal of Theoretical Physics*, Vol. 3, No. 4, p. 295.

Kademova, K. V. and Kálnay, A. J. (1970). *International Journal of Theoretical Physics*, Vol. 3, No. 2, p. 115.

Kademova, K. V. and Kraev, M. M. (1970). Realisations of the Representations of para-Fermi Algebra—Part III: Fock Space of Fermi Operators. *International Journal of Theoretical Physics*, Vol. 3, No. 3, p. 185.

Kademova, K. V. and Palev, T. D. (1970). Lie Algebras and Quasifield Operators. *International Journal of Theoretical Physics*, Vol. 3, No. 5, p. 337.